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Separation of the Massless Spin-1 Equation in Robertson–Walker Space-Time

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The massless spin-1 free field equation is studied via the Newman–Penrose formalism and separated by the Chandrasekhar–Teukolski method. The temporal and angular equations are explicitly integrated. The radial equations are solved in the flat-universe case. The closed-universe case shows, in principle, the existence of a discrete spectrum of the energy of the massless particles.

1. INTRODUCTION

It is well known that the massless free field equation can formally be written in curved space-time in a general way for arbitrary spin. In the context of the Newman and Penrose (1962) formalism this is done in terms of the equation (Penrose, 1965)

$$\nabla_{AA'}\phi^{A}_{BC\ldots L} = 0, \qquad \phi_{ABC\ldots L} = \phi_{(ABC\ldots L)} \tag{1}$$

The study of equation (1) is of interest because it involves as particular situations the Bianchi identity in empty space ($s = 2$, gravitons), the source-free Maxwell equations ($s = 1$), and the Dirac–Weyl equation for the neutrino ($s = 1/2$) (Penrose and Rindler, 1986).

The formulation (1) is consistent in general for $s = 1/2$ and $s = 1$. It is inconsistent already in Minkowski space in the case of electromagnetic interaction (Fierz and Pauli, 1939) and in general for $s > 1$ unless the space-time is conformally flat (Buchdahl, 1958, 1962); Wünsch, 1978; Penrose and Rindler, 1986). A possible way to overcome these difficulties in a uniform manner has been proposed in terms of symmetrized field equations (Buchdahl, 1982; Wünsch, 1985; Illge, 1988, 1992, 1993).

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The solution of equation (1) can be explicitly given in a conformally flat space-time for arbitrary spin (Penrose and MacCallum, 1972; Penrose and Rindler, 1986) and it was just the study of this solution that motivated the development of the twistor formalism (Penrose, 1968).

In the present paper we study explicitly equation (1) for spin $s = 1$ in the case of the Robertson–Walker space-time. The solution of equation (1) we propose is different from the mentioned general one for arbitrary spin when specialized to the present case. By applying a method similar to that used by Chandrasekhar to solve the Dirac equation in the Kerr metric (Chandrasekhar, 1983), we are able to completely separate in equation (1) the temporal and angular dependences of the wave function for $s = 1$. The resulting time equation can be formally integrated. The integration, however, depends on the knowledge of the dynamics of the cosmological background. Also, the angular equations are integrated, since they reduce to known equations of mathematical physics. The radial equations are explicitly integrated in the flat-universe case.

The final angular equations turn out to be independent of the space curvature of space-time, namely of whether the universe is closed, flat, or open, while the radial equations do strictly depend on this property. In particular, the final radial equations of the closed-universe case support the existence of discrete values of the integration constant relative to the separation of the time dependence. By analogy with the neutrino case (Montaldi and Zecca, 1994), we interpret this constant as the energy of the massless particle of the field.

2. FORMULATION IN THE ROBERTSON–WALKER GEOMETRY

We take equation (1) explicitly for $s = 1$ and in the case of the Robertson–Walker metric of the form

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1 - ar^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \tag{2}$$

We apply to it the Newman–Penrose formalism (Newman and Penrose, 1962) by adopting Chandrasekhar's notations and mathematical conventions. We choose as null tetrad frame $\{l^i, n^i, m^i, m^{*i}\}$ the one whose associated directional derivatives are given by

$$D \equiv l^i\partial_i = \frac{1}{\sqrt{2}}\,[\partial_t + R^{-1}(1 - ar^2)^{1/2}\partial_r]$$

$$\Delta \equiv n^i\partial_i = \frac{1}{\sqrt{2}}\,[\partial_t - R^{-1}(1 - ar^2)^{1/2}\partial_r]$$

$$\delta \equiv m^i\partial_i = \frac{1}{\sqrt{2}rR}(\partial_\theta + i \csc\theta\partial_\phi)$$

$$\delta^* \equiv m^{*i}\partial_i = \frac{1}{\sqrt{2}rR}(\partial_\theta - i \csc\theta\partial_\phi) \tag{3}$$

and whose corresponding nonzero spin coefficients are given by (Montaldi and Zecca, 1994)

$$\rho = -\frac{1}{\sqrt{2}rR}[r\dot{R} + (1 - ar^2)^{1/2}], \qquad \beta = -\alpha = \frac{\cot\theta}{2\sqrt{2}rR}$$

$$\mu = \frac{1}{\sqrt{2}rR}[r\dot{R} - (1 - ar^2)^{1/2}], \qquad \epsilon = -\gamma = \frac{\dot{R}}{2\sqrt{2}R} \tag{4}$$

By using in equation (1) the relation $\nabla_{AX'} = \sigma^a_{AX'}\nabla_a$ and by making explicit the expressions of the covariant derivatives in terms of the tabulated spin coefficients (Chandrasekhar, 1983; Penrose and Rindler, 1986) and with the usual identifications $D \equiv \partial_{00'}$, $\delta \equiv \partial_{01'}$, $\delta^* \equiv \partial_{10'}$, $\Delta \equiv \partial_{11'}$, we find the equations

$$\begin{aligned}(D - \rho)\phi_{10} - (\delta^* - 2\alpha)\phi_{00} - \rho\phi_{01} &= 0\\ (D - \rho + 2\epsilon)\phi_{11} + \delta^*\phi_{01} &= 0\\ \delta\phi_{10} - (\Delta - 2\gamma + \mu)\phi_{00} &= 0\\ (\delta + 2\beta)\phi_{11} - (\Delta + \mu)\phi_{10} - \mu\phi_{01} &= 0\end{aligned} \tag{5}$$

which must be solved for the symmetric spinor $\phi_{AB} = \phi_{BA}$.

3. SEPARATION OF THE EQUATIONS

The ϕ dependence can be directly separated in equations (5) by the substitution $\phi_{AB} \to \phi_{AB}\exp(im\phi)$ ($m = 0, \pm 1, \pm 2, \ldots$) to get

$$\begin{aligned}rR\sqrt{2}(D - 2\rho)\phi_{10} - L_1^-\phi_{00} &= 0\\ rR\sqrt{2}(D - \rho + 2\epsilon)\phi_{11} - L_0^-\phi_{10} &= 0\\ -rR\sqrt{2}(\Delta + \mu - 2\gamma)\phi_{00} + L_0^+\phi_{10} &= 0\\ -rR\sqrt{2}(\Delta + 2\mu)\phi_{10} + L_1^+\phi_{11} &= 0\end{aligned} \tag{6}$$

where

$$L_n^{\pm} = \partial_\theta \mp m\csc\theta + n\cot\theta \tag{7}$$

and ϕ_{AB} depends now on the variables r, θ, t. By setting

$$\begin{aligned}\phi_{00} &= \phi_0(r)S_0(\theta)T(t)\\ \phi_{01} &= \phi_{10} = \phi_1(r)S_1(\theta)T(t)\\ \phi_{11} &= \phi_2(r)S_2(\theta)T(t)\end{aligned} \tag{8}$$

in equations (6) the separation of the θ dependence gives the equations

$$\begin{aligned}L_1^- S_0 &= \lambda_1 S_1, \qquad L_0^- S_1 = \lambda_2 S_2\\ L_0^+ S_1 &= \lambda_3 S_0, \qquad L_1^+ S_2 = \lambda_4 S_1\end{aligned} \tag{9}$$

$\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the relative separation constants. The corresponding surviving equations in the r, t variables can be further separated, with separation constant ik ($k \in R$), giving

$$\begin{aligned}ik &= -\frac{\dot{T}}{T}R - 2\dot{R}\\ ik &= (1 - ar^2)^{1/2}\frac{\phi_1'}{\phi_1} + \frac{2}{r}(1 - ar^2)^{1/2} - \frac{\lambda_1}{r}\frac{\phi_0}{\phi_1}\\ ik &= (1 - ar^2)^{1/2}\frac{\phi_2'}{\phi_2} + \frac{1}{r}(1 - ar^2)^{1/2} - \frac{\lambda_2}{r}\frac{\phi_1}{\phi_2}\\ ik &= -(1 - ar^2)^{1/2}\frac{\phi_0'}{\phi_0} - \frac{1}{r}(1 - ar^2)^{1/2} - \frac{\lambda_3}{r}\frac{\phi_1}{\phi_0}\\ ik &= -(1 - ar^2)^{1/2}\frac{\phi_1'}{\phi_1} - \frac{2}{r}(1 - ar^2)^{1/2} - \frac{\lambda_4}{r}\frac{\phi_2}{\phi_1}\end{aligned} \tag{10}$$

where use has been made of the explicit form (3), (4) of the directional derivatives and of the spin coefficients, respectively.

The integration of the time equation gives

$$T(t) = T(0)\frac{R^2(0)}{R^2(t)}\exp\left[-ik\int_0^t \frac{dt'}{R(t')}\right] \tag{11}$$

which depends on the particular choice of the underlying cosmological model which determines the structure of the function $R(t)$. Instead, the r and θ equations are independent of such a choice.

4. THE ANGULAR EQUATIONS

Equations (9) imply four second-order equations each involving only one function among S_0, S_1, and S_2 plus two coupled second-order equations in the pair of functions S_0, S_2. However, by taking into account that

$$L_1^+ L_0^- = L_1^- L_0^+ \tag{12}$$

as can be directly checked from the definition (7), by assuming the condition

$$\lambda_1\lambda_3 = \lambda_2\lambda_4 = -\lambda^2 \tag{13}$$

one is left with the three independent equations

$$L_1^- L_0^+ S_1 = -\lambda^2 S_1 \tag{14a}$$

$$L_0^- L_1^+ S_2 = -\lambda^2 S_2 \tag{14b}$$

$$L_0^+ L_1^- S_0 = -\lambda^2 S_0 \tag{14c}$$

With regard to the integration, we remark that equation (14b) gives equation (14c) after the substitution $m \to -m$.

We are therefore looking for solutions of equations (14a) and (14b) that are regular in $\theta = 0$ and $\theta = \pi$. By using the explicit expression (8) and then by setting $\xi = \cos\theta$, we find that equation (14a) becomes

$$S_1'' + \frac{2\xi}{\xi^2 - 1} S_1' + \frac{\lambda^2(1 - \xi^2) - m^2}{(1 - \xi^2)^2} S_1 = 0 \tag{15}$$

whose acceptable solutions corresponding to $\lambda^2 = l_1(l_1 + 1)$ are the associated Legendre functions (Abramovitz and Stegun, 1970)

$$S_1(\theta) = (1 - \xi^2)^{|m|/2} P_{l_1}^m(\xi), \qquad l_1 = |m|, |m| + 1, |m| + 2, \ldots \tag{16}$$

With regard to equation (14b), after using equation (7), it takes the form

$$S_2'' + \cot\theta\, S_2' + \left(\frac{2m\cos\theta - 1 - m^2}{\sin^2\theta} + \lambda^2\right) S_2 = 0 \tag{17}$$

For its solution let us first consider the case $m \geq 1$. By setting

$$S_2 = (1 - \xi)^{(m-1)/2}(1 + \xi)^{(m+1)/2} f_2(\xi), \qquad \xi = \cos\theta \tag{18}$$

one finds for f_2 the equation

$$(1 - \xi^2) f_2'' + 2[1 - \xi(m + 1)] f_2' + [\lambda^2 - m(m + 1)] f_2 = 0 \tag{19}$$

whose acceptable solutions corresponding to $\lambda^2 = l_2(l_2 + 1)$ can be written in terms of Jacobi polynomials (Abramovitz and Stegun, 1970) so that

$$S_{2,l_2,m} = (1 - \cos\theta)^{(m-1)/2}(1 + \cos\theta)^{(m+1)/2} P_{l_2-m}^{(m+1,m-1)}(\cos\theta) \tag{20}$$

$$m \geq 1, \quad l_2 = m, m + 1, m + 2, \ldots$$

If $m \leq -1$, it suffices to replace [see equation (17)] m by $|m|$ and ξ by $-\xi$.

Thus, apart from an irrelevant factor [recall that $P_n^{(\alpha,\beta)}(-x) = (-)^n P_n^{(\beta,\alpha)}(x)$],

$$S_{2,l_2,m} = (1 + \cos\theta)^{(|m|-1)/2}(1 - \cos\theta)^{(|m|+1)/2}P_{l_2-|m|}^{(|m|-1,|m|+1)}(\cos\theta)$$

$$m \le -1, \quad l_2 = |m|, |m| + 1, |m| + 2, \ldots \tag{21}$$

If now $m = 0$, by setting

$$S_2 = (1 - \xi^2)^{1/2} f_2(\xi), \qquad \xi = \cos\theta \tag{22}$$

in equation (17), we find in a similar way

$$S_{2,l_2,0}(\theta) = \sin\theta\, P_{l_2+2}^{(1,1)}(\cos\theta), \qquad l_2 = 0, 1, 2, 3, \ldots \tag{23}$$

which corresponds here to $\lambda^2 = (l_2 + 1)(l_2 + 2)$.

Finally, by combining the results relative to all the angular equations, we see that the possible values of λ^2 are of the form $\lambda^2 = l(l + 1)$, $l = 1, 2, 3, \ldots$.

5. THE RADIAL EQUATIONS

Defining the operators

$$A_b = (1 - ar^2)^{1/2}\frac{d}{dr} + \frac{b}{r}(1 - ar^2)^{1/2} - ik \qquad (b \in R) \tag{24}$$

which can be easily shown to have the properties

$$A_b^* A_b = A_b A_b^*, \qquad A_1^* r A_2 = A_1 r A_2^* \tag{25}$$

we can write equations (10) compactly as

$$A_2\phi_1 = \frac{\lambda_1}{r}\phi_0, \qquad A_1\phi_2 = \frac{\lambda_2}{r}\phi_1$$

$$A_1^*\phi_0 = \frac{-\lambda_3}{r}\phi_1, \qquad A_2^*\phi_1 = \frac{-\lambda_4}{r}\phi_2 \tag{26}$$

Here six second-order equations can be derived from equations (26) that must be satisfied by the functions ϕ_0, ϕ_1, ϕ_2. However, by using equation (25) and the condition (13), one can easily check that the solution of the equations is reduced to the study of the three independent equations

$$A_4 r A_1^*\phi_0 = \frac{\lambda^2}{r}\phi_0 \tag{27a}$$

$$A_1^* r A_2\phi_1 = \frac{\lambda^2}{r}\phi_1 \tag{27b}$$

$$A_2^* r A_1 \phi_2 = \frac{\lambda^2}{r} \phi_2 \tag{27c}$$

Furthermore, if ϕ_0 satisfies equation (27a), then, by taking the complex conjugate equation, we have that ϕ_0^* satisfies equation (27c). Therefore we can confine ourselves to looking for solutions of equations (27a) and (27b). By using the definition (24), we find that equations (27a) and (27b) become, respectively,

$$r(1 - ar^2)\phi_0'' + \phi_0'(4 - 5ar^2) + \phi_0\left(k^2 r - 3ar + \frac{2 - \lambda^2}{r} + 2ik(1 - ar^2)^{1/2}\right) = 0 \tag{28a}$$

$$r(1 - ar^2)\phi_1'' + \phi_1'(4 - 5ar^2) + \phi_1\left(k^2 r - 4ar + \frac{2 - \lambda^2}{r}\right) = 0 \tag{28b}$$

which both fall into the class of Fuchs equations (e.g., Moon and Spencer, 1961) and have therefore the behavior

$$\phi_d(r) = r^{[(4\lambda^2+1)^{1/2}-3]/2} f_d(r, k^2), \qquad d = 0, 1 \tag{29}$$

near $r = 0$, f_d being there a regular function. If one assumes the conditions $\phi_0(0) = \phi_1(0) = 0$ which naively follows from the nature of the radial equations, then the acceptable solutions ϕ_d are those for which $4\lambda^2 + 1 > 9$ or $\lambda^2 = l(l + 1)$ with now $l = 2, 3, 4, \ldots$.

The **case $a = 0$** is the only case in which we are able to solve equations (28) completely. Indeed, by setting

$$\phi_d(r) = r^{[(4\lambda^2+1)^{1/2}-3]/2} \exp(-ikr)\, Z_d(r), \qquad d = 0, 1 \tag{30}$$

in equations (28a) and (28b) and then $\xi = 2ikr$ in the resulting equations, we get the confluent hypergeometric equations

$$\xi Z_d'' + [1 + (4\lambda^2 + 1)^{1/2} - \xi] Z_d' - \frac{1}{2}[(4\lambda^2 + 1)^{1/2} + 2d - 1] Z_d = 0, \qquad d = 0, 1 \tag{31}$$

Therefore we have

$$Z_d(r) = \phi\left(\frac{1}{2}(4\lambda^2 + 1)^{1/2} + d + \frac{1}{2};\ 1 + (4\lambda^2 + 1)^{1/2};\ 2ikr\right) \qquad (d = 0, 1)$$

for the acceptable solutions.

Finally, the **case $a = 1$** is of particular interest. The solutions of equations (28) are subject in this case to the additional constraints $\phi_d(1) = 0$, $d = 0$, 1, which follow from equations (10), λ, k being independent constants. Equivalently, from equation (29), we have

$$f_d(1, k^2) = 0, \qquad d = 0, 1 \tag{32}$$

In principle both the equation $f_0(1, k^2) = 0$ and $f_1(1, k^2) = 0$ are satisfied by a discrete set of values of k^2. It is an open question to establish whether the problem has a solution, namely of determining the k^2 common elements.

6. CONCLUSIONS

The method employed in the previous sections, which is an extension of the Chandrasekhar–Teukolski method used in the separation of the Dirac equation in the Kerr geometry (e.g., Chandrasekhar, 1983), involves the consideration of the two separation constants λ^2 and k. The constant λ^2 represents the eigenvalue of the angular equations, as is evident from equations (14). The solutions of these equations are the extension to the spin-1 case of those relative to the Dirac equation both in the Kerr metric (Chandrasekhar, 1983) as well as the Robertson–Walker metric (Montaldi and Zecca, 1994; Zecca, 1995). With regard to the constant k, we interpret k^2 to give the energy of the massless particle. This is supported by the structure of the time solution (11) even if the radial equations (10) do not have the explicit form of a Schrödinger-like eigenvalue problem, as directly follows for the Dirac equation in the Kerr geometry case (Chandrasekhar, 1983) or for the neutrino in the Robertson–Walker metric (Montaldi and Zecca, 1994). According to this interpretation, it is of interest that the closed-universe case admits, in principle, the existence of discrete values of the energy of the particles, a property already tested both for the neutrino and the electron of the Dirac equation in the Robertson–Walker metric (Montaldi and Zecca, 1994; Zecca, 1995).

By coming back to the general case, the problem is open whether equation (1) can be separated, by means of the Chandrasekhar method, in the Robertson–Walker metric also for values of the spin $s > 1$.

Finally, according to the previous interpretation of k^2, it is also an open problem whether the discrete spectrum of the energy of the particle is a general property that holds in the closed-universe case for arbitrary spin.

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